A Puzzle to Puzzle You

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Question

In this question, we discuss one of the most popular puzzles considered by mathematicians - the $n^2 - 1$ puzzle.

Well, the logic is simple. It is a sliding puzzle having $n^2 - 1$ square tiles numbered 1 to $n^2 - 1$ in a frame that is n tiles high and n tiles wide, leaving one unoccupied tile position. Tiles in the same row or column of the open position can be moved by sliding them horizontally or vertically, respectively. The goal of the puzzle is to place the tiles in numerical order.

You might have already seen that for $n = 4$, the puzzle is not always solvable and that the famous "puzzler" Sam Loyd (who claimed to have "invented" it) used and monetized it very cleverly - by posing whether an impossible case can be solved (though, people are not sure whether he knew it was solvable or not)

Here is the case. Now for the question

(a) Given any $n \geq 3$, find an "unsolvable" case with proof.

(b) For $n = 3$, device an algorithm to solve the puzzle if it is solvable.

(c) Generalise the process described above to show that if the puzzle is solvable, you can solve it in finitely many steps no matter what n is! (That is, give an algorithm).

(d) For every $n \geq 3$, if the tiles are unlabelled and labelled again uniformly at random, what is the probability that the puzzle is solvable? (Not that if a is true, then this probability is neither 0 nor 1)

(e) The grand finale!! - The game can be generalised to $mn-1$ puzzle with $m, n \geq 3$, what can you say about solvability, probability of solution and and algorithm to solve in such a case?

Just for fun - here is a simulator of the puzzle for the lower dimensions. Try it out to check your algorithm!

https://www.helpfulgames.com/subjects/brain-training/sliding-puzzle.html