Session 10

20/07/25

Remark: (X_t) on (T, d)

$d \to$ was a proxy for dependence

☆ L.C:

$|X_t - X_s| \leqslant C\, d(s,t)$ a.s.

S.G.P.

$\mathbb{E}\, e^{\lambda (X_s - X_t)} \leqslant e^{\lambda^2 d(s,t)^2/2}$

$\Updownarrow$

$\mathbb{P}\big(|X_s - X_t| > t\, d(s,t)\big) \leqslant e^{-t^2/2}$

in p

Random matrices

$X_x = \|Ax\|$

eg. Gaussian Process: $(X_t)_{t \in T}$ jointly Gaussian, i.e., $\forall\, a_1, \dots, a_k$

$\sum_{i=1}^{k} a_i X_{t_i}$ is Gaussian.

- Assume $\mathbb{E} X_t = 0 \quad \forall t$.

Define $d(s,t) := \left[\mathbb{E}\,(X_s - X_t)^2\right]$ — this defines a metric on T

$$\mathbb{E}\, e^{\lambda (X_s - X_t)} = e^{\lambda^2 d(s,t)^2/2} \quad \longrightarrow \text{ sG P holds.}$$

Covering method $\longrightarrow$ Chaining $\longrightarrow$ Generic Chaining $\to (\cdot)$

Optimal for Gaussian Processes

Random Matrices

- Measure on Ω is a set f^{n} $\mu : \mathcal{B} \longrightarrow [0,\infty)$, where $\mathcal{B} \subseteq \mathcal{P}(\Omega) = 2^{\Omega}$
  i) $A \subset B$, $\mu(A) \leq \mu(B)$
  ii) $\mu(A \amalg B) = \mu(A) + \mu(B)$.

eg. Dirac measures: $\delta_c(A) = \begin{cases} 1 & \text{if } c \in A \\ 0 & \text{o.w.} \end{cases}$

$$\delta_c(\{c\}) = 1$$

Prob. ms. on $\mathbb{R}$: $\longleftrightarrow$ A r.v.

$$X \longleftrightarrow \mu_X \quad , \quad \mu_X(A) = \mathbb{P}(X \in A)$$
$$F_X(x) = \mu(-\infty, x]$$

- $X = c$ a.s. $\Rightarrow \mu_X(A) = \mathbb{P}(X \in A) = \begin{cases} 1 & \text{if } c \in A \\ 0 & \text{o.w.} \end{cases} = \delta_c$

- $$X = \begin{cases} c_1 & \text{wp } p \\ c_2 & \text{wp } q \end{cases} \Rightarrow \mu_X(A) = \begin{cases} 1 & c_1, c_2 \in A \\ p & c_1 \in A,\ c_2 \notin A \\ q & c_2 \in A,\ c_1 \notin A \\ 0 & c_1, c_2 \notin A \end{cases}$$
  $\mu_X(A) = \underline{\mathbb{P}(X \in A)}$

  deterministic

$$\boxed{\mu_X = p\,\delta_{c_1} + q\,\delta_{c_2}}$$

$(f_1 + f_2)(x)$
$= f_1(x) + f_2(x)$

$$\mu_X(A) = p\,\delta_{c_1}(A) + q\,\delta_{c_2}(A)$$
$$= p\,\mathbb{1}_{c_1 \in A} + q\,\mathbb{1}_{c_2 \in A}$$

X discrete, takes values x_i wp p_i, then

$$\boxed{\mu_X = \sum_i p_i \delta_{x_i}}$$

$$\mu_X(A) = \sum_i p_i \delta_{x_i}(A) = \sum_{x_i \in A} p_i = \mathbb{P}(X \in A)$$

Weak convergence of measures:

- $X_n \xrightarrow{d} X \overset{d}{\iff} F_{X_n} \longrightarrow F_X$ ptwise on continuity pts of F_X

(Levy Cont. Thm) $\iff \mathbb{E} f(X_n) \longrightarrow \mathbb{E} f(X) \quad \forall f \in C_b(\mathbb{R})$

$\iff \phi_{X_n} \longrightarrow \phi_X$ ptwise $\quad (\phi_X(t) := \mathbb{E} e^{itX})$

- Weak conv. of m.s. :

  - X - r.v. and μ_X - corresponding measure

$$\int f \, d\mu_X := \mathbb{E}[f(X)]$$

$$\therefore \int f \, d\delta_c := \mathbb{E}[f(X)] \qquad X = c \text{ a.s.}$$
$$= f(c)$$

$$\int f \, d\Big(\sum_i p_i \delta_{x_i}\Big) = \sum_i p_i \Big(\int f \, d\delta_{x_i}\Big) = \sum p_i f(c_i)$$
$$= \mathbb{E}[f(X)]$$

$X = c_i$ w.p. p_i

<u>Defn (w.c.)</u> : $\mu_n \Rightarrow \mu$ if $\int f \, d\mu_n \longrightarrow \int f \, d\mu \quad \forall f \in C_b(\mathbb{R})$

p ms on $\mathbb{R}$

$X_n \xrightarrow{d} X \qquad \mathbb{E} f(X_n) \longrightarrow \mathbb{E} f(X)$

Random Matrix Theory

★. **Wigner Matrices:** $\quad W_n := (w_{ij})_{n\times n}$ Hermitian

$$w_{ij} = \overline{w_{ji}} \overset{iid}{\sim} \underline{\xi} \quad \forall\, i>j$$

$$w_{kk} \underset{iid}{\sim} \underline{\xi_d}$$

$W_1, W_2, W_3 \cdots$ arising out of a minor process.

$$W_2 = \left\{ \begin{matrix} W_1 = w_{1,1} & w_{1,2} \\ w_{2,1} & w_{2,2} \end{matrix} \right. \qquad \left.\begin{matrix} w_{1,1} & w_{1,2} & w_{1,3} & \cdots \\ w_{2,1} & w_{2,2} & w_{2,3} & \cdots \\ w_{3,1} & w_{3,2} & w_{3,3} & \cdots \\ \vdots & & & \ddots \end{matrix}\right\} = W_3$$

$$\mathbb{E}\,\xi = 0, \qquad \mathrm{Var}\,\xi = 1$$

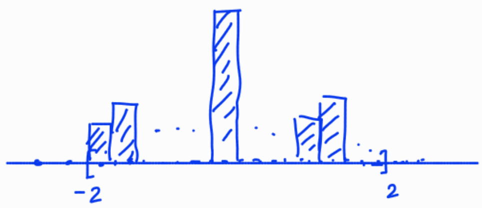
- We are interested in eigenvalues, eigenvectors, sing. values of $(W_n)_n$.

- $$\sum_{i=1}^{n} \lambda_i^2(W_n) = \mathrm{Tr}(W_n^* W_n) \leftarrow$$

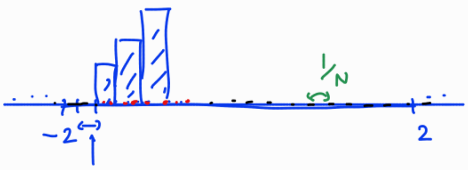
We need to scale by $\sqrt{n}$.

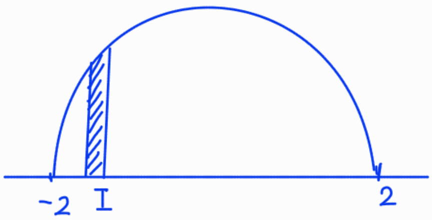
$$\mathbb{E}\left[\mathrm{Tr}(W_n^* W_n)\right] = \mathbb{E}\left[\sum_{i,j} |w_{ij}|^2\right] = n$$

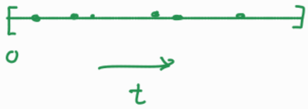
- We consider $M_n := \dfrac{W_n}{\sqrt{n}}$, $\lambda_i \equiv \lambda_i(M_n)$.



∅ Semicircle Law (SCL) : (Wigner) $\rho_{sc}(x) = \frac{1}{2\pi}\sqrt{(4-x^2)_+}$







$M_n \longrightarrow \lambda_1(M_n) \geq \lambda_2(M_n) \geq \cdots \geq \lambda_n(M_n)$

ESD – Empirical Spectral distribution

Define $\mu_n := \frac{1}{n}\sum_{i=1}^{n} \delta_{\lambda_i(M_n)}$ – random measure

$$\mu_n(I) = \frac{\#\text{ eigenvalues in the interval } I}{n}$$

- For every interval I, $\mu_n(I) \longrightarrow \rho_{sc}(I) = \int_I \rho_{sc}(x)\,dx$. a.s.

$\iff \mu_n(-\infty, a) \longrightarrow \rho_{sc}(-\infty, a]$ a.s. $\forall a \in \mathbb{R}$

$\iff \int f\,d\mu_n \longrightarrow \int f\,d\rho$ a.s. $\forall f \in C_b(\mathbb{R})$

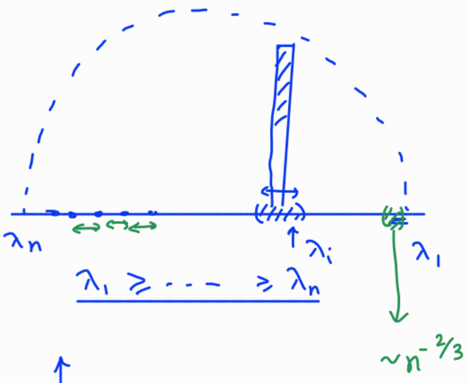
① Sample $\underbrace{(M_n)_n}$ ② ESD $(\mu_n)_n$ ③ $\mu_n \Longrightarrow \rho$ a.s.

Only randomness in here.

① Thm: (SCL) If $M_n = \frac{W_n}{\sqrt{n}}$, W_n is obtained from a Wigner minor process as before, then the ESD $\mu_n := \frac{1}{n}\sum_i \delta_{\lambda_i(M_n)}$ satisfies

$$\mu_n \Longrightarrow \rho_{sc} \quad \text{a.s.}$$

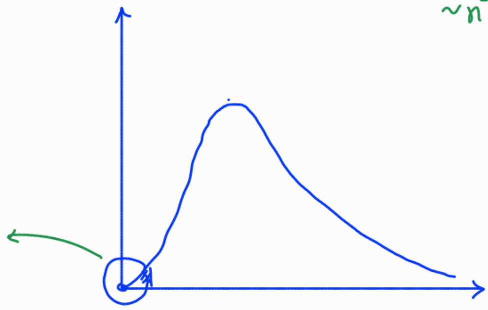
2. Wigner - Dyson - Mehta Conjecture.



$$\rho(\lambda_i) N (\lambda_i - \lambda_{i+1}) \xrightarrow{d} \underset{\substack{\top \\ \text{density} \\ \propto x^2 e^{-\frac{c}{\pi} x^2}}}{RV_1}$$

— Empirical process of Eigvals is strongly correlated.

Eigenvalue repulsion !



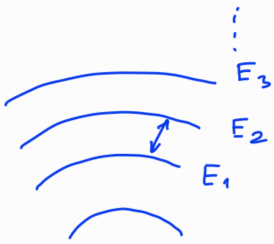
⊘ Formulated ~ 1960
Proven $\sim$ 2010's Tao - Vu , Erdös - Schlein - ....

$$\rho(\lambda_i) N |\lambda_i - \lambda_{i+1}| \xrightarrow{d} RV_{\mathbb{R}}$$
$$\rho(\lambda_i) N |\lambda_i - \lambda_{i+1}| \xrightarrow{d} RV_{\mathbb{C}}$$

- Very universal
- Very robust etc.

Motivation: $\underset{\substack{| \\ \text{operator}}}{\hat{H}} \psi = \underset{\substack{\uparrow \\ \mathbb{R}}}{E} \psi$ - Eigenvalue eq$\underline{n}$.

Energy levels are the eigenvalues of $\hat{H}$.



- For large atom, $\hat{H}$ is incomputable.

- $\lambda_i - \lambda_{i+1}$.

Three classical thms in $\mathbb{P}$: $X_1, \dots, X_n, \dots$ iid X, $X \in L^2$

1. LLN: $\frac{1}{n}\sum_{i=1}^n X_i \xrightarrow{a.s.} \mathbb{E}X$ ✓ $\mu_n = \frac{1}{n}\sum \delta_{\lambda_i(M)} \Rightarrow \rho_{sc}$ a.s.

2. CLT: $\underbrace{\sqrt{n}}\left(\frac{1}{n}\sum_{i=1}^n X_i - \mathbb{E}X\right) \longrightarrow N(0, Var(X))$

3. Maximum: $a_n\left(\max_n X_n - b_n\right) \longrightarrow \mathcal{G}$ (Gumbel distr$^{\underline{n}}$)
(under mild assumption)

② Fluctuations:

random measure

$$\frac{1}{n}\sum_i \delta_{\lambda_i} \Longrightarrow \rho_{sc} \quad a.s.$$

random variable

$$\Longleftrightarrow \int f \, d\left(\frac{1}{n}\sum \delta_{\lambda_i}\right) \longrightarrow \int f \rho_{sc}. \quad a.s.$$

$$\Longleftrightarrow \boxed{\frac{1}{n}\sum_{i=1}^n f(\lambda_i) \xrightarrow{a.s.} \int f \, d\rho_{sc}} \quad \forall f \in \underline{\underline{C_b}}(\mathbb{R})$$

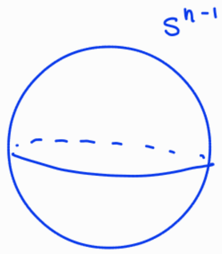
$$\varnothing \quad \underbrace{n}\left(\frac{1}{n}\sum_{i=1}^n f(\lambda_i) - \int f \, d\rho_{sc}\right) \xrightarrow{d} N(0, v_f) \quad \forall f \in C^3(\mathbb{R})$$

(Fluctuations)

$$V_f = \frac{1}{2\pi^2}\iint_{\mathbb{R}^2}\left(\frac{f(x) - f(y)}{x - y}\right)^2 \frac{4 - xy}{\sqrt{4 - x^2}\sqrt{4 - y^2}} \, dx \, dy$$

Much faster than iid CLT.

$\varnothing$ Eigenvectors: $M_n \longrightarrow \lambda_1 \geq \dots \geq \lambda_n$
$u_1 \geq \dots \geq u_n$



" $u_1, u_2, \dots, u_n$ behave as if they were sample u.a.r. from S^{n-1} "

★ Quantum Unique Ergodicity (Math.) = Eigenstate Thermalisation Hyp. (Phys.)

[~ 2018]

i) $u_i^{(1)} \sim \frac{1}{\sqrt{n}}$ (Eigvec. delocalisation)

ii) $\langle u_i, u_j \rangle = 0$

$$\langle u_i, A u_j \rangle = \langle u_i, \underbrace{(A - \tfrac{1}{n}\mathrm{Tr}(A))}_{\mathring{A}} u_j \rangle + \tfrac{1}{n}\mathrm{Tr}(A)\langle u_i, u_j \rangle$$

A deterministic

$$= \delta_{ij} \underbrace{\langle A \rangle}_{= \frac{1}{n}\mathrm{Tr}} + \underbrace{\langle u_i, \mathring{A} u_j \rangle}$$

$$\langle u_i, \mathring{A} u_j \rangle \longrightarrow 0 \quad \text{as } n \to \infty \quad \forall\, i \neq j$$

$$\sqrt{n}\, \langle u_i, \mathring{A} u_j \rangle \longrightarrow N(0, \langle \mathring{A}^2 \rangle) \qquad \text{[ETH / QUE]}$$

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i.i.d. matrices : $X_n = (x_{ij})_{n \times n}$, $x_{ij} \overset{iid}{\sim} \xi$, $\mathbb{E}\xi = 0$, $\mathrm{Var}\,\xi = 1$

$\mathbb{E}|\xi|^2 = 1$

$\mathbb{E}\,\xi^2 = 0$ (if $\mathbb{C}$).

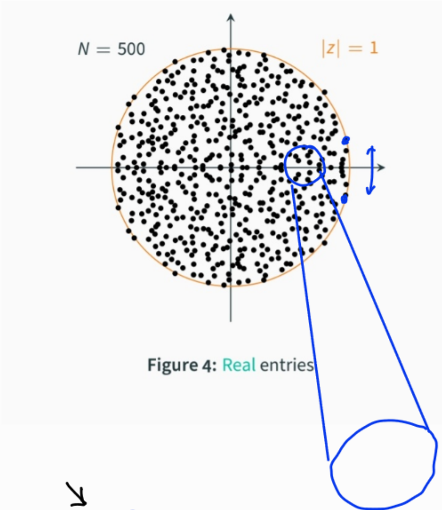

Figure 4: Real entries

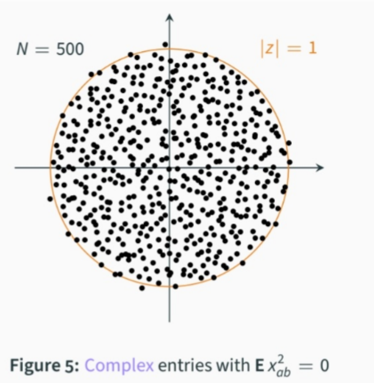

Figure 5: Complex entries with $\mathbf{E}\, x_{ab}^2 = 0$

'SCL $\longrightarrow$ Circular law

① $\frac{1}{n}\sum_{i=1}^{n} \delta_{\lambda_i\left(\frac{X_n}{\sqrt{n}}\right)} \Rightarrow \rho_c$ a.s.

(ESD)

Unif dist. on unit circle.

② Bulk Universality ✓ Edge Univ ✓

③ Eigenvector deloc ✓ (?) 'Spectrum is very unstable'

④ ETH / QUE. (Open)

Tools and Techniques:

Hermitian:

SCL: 1) Method of moments

2) * Stieltjes Transform method → much more robust.

/ Green's fn method → Local SCL as well

Univ. of Gaps 3) Green's fn comparison theorem (Tao, Vu)

Eigenvec: 4) Dyson Brownian Motion.

Non Hermitian:

(*) Girko's Hermitisation Technique → Requires bounds on the smallest singular value....

$$NH \xrightarrow{\text{GHT}} H$$