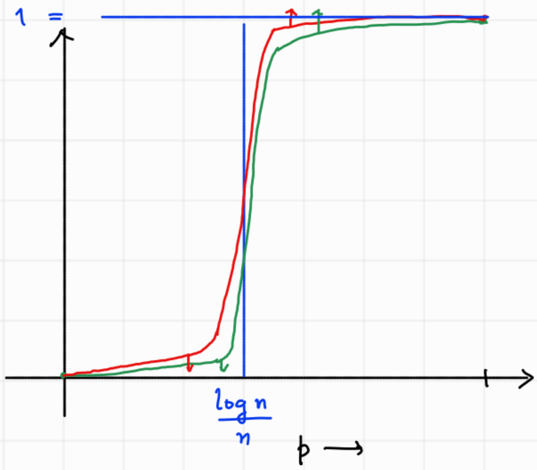
Session - 2

21/05/2025

- $G(n,p)$

$$\mathbb{P}(G(n,p) \text{ has no isolated vertices}) = \begin{cases} o_{c_1}(1) & p < c_1 \frac{\log n}{n}, \ c_1 < 1 \\ 1 - o_{c_2}(1) & p > c_2 \frac{\log n}{n}, \ c_2 > 1 \end{cases}$$

$$\mathbb{P}(G(n,p) \text{ is connected}) = \begin{cases} o_{c_1}(1) & p < c_1 \frac{\log n}{n}, \ c_1 < 1 \\ 1 - o_{c_2}(1) & p > c_2 \frac{\log n}{n}, \ c_2 > 1 \end{cases}$$



n - fixed

- As $n \to \infty$, the red and green curves flatt

The properties of {Having no isolated pts} & {being connected} for $G(n,p)$ has a threshold at $p = \frac{\log n}{n}$

$$\boxed{\mathbb{P}(|X - \mathbb{E}X| \geq c) \leq o(1)}$$

$$X = \sum_{i=1}^{n} X_i$$
$$c = c(n)$$

§. Exponential tail bounds

eg. $Z \sim N(0,1)$, $M_Z(s) = \mathbb{E} e^{sZ}$ ← moment gen $f^{\underline{n}}$ (MGF) or Laplace Transform

$$M_Z(s) = e^{s^2/2}$$

$Z \sim N(0, \sigma^2)$, σZ_1

$$M_Z(s) = e^{s^2\sigma^2/2}$$

$sG(\sigma_i^2) \subseteq sG(\sigma_j^2)$

Sub Gaussian Random Variables:

① **Def^n:** X is said to be sub-Gaussian ($sG(\sigma^2)$) if the mgf of $X - \mathbb{E}X$ exists and

$$\mathbb{E}e^{s(X-\mathbb{E}X)} \leqslant e^{s^2\sigma^2/2}$$

The mgf is dominated by that of a $N(0,\sigma^2)$.

② **Tail Probabilities:**

$$\mathbb{P}(X - \mathbb{E}X \geqslant \beta), \quad \beta > 0$$

$$= \mathbb{P}\left(e^{\lambda(X-\mathbb{E}X)} \geqslant e^{\lambda\beta}\right) \quad \forall \lambda > 0$$

$$\leqslant e^{-\lambda\beta}\,\mathbb{E}e^{\lambda(X-\mathbb{E}X)} \leqslant e^{-\lambda\beta + \frac{\lambda^2\sigma^2}{2}} \quad \forall \lambda > 0$$

Optimizing λ: $\lambda^* = \frac{\beta}{\sigma^2}$, we see that

$$\mathbb{P}(X - \mathbb{E}X \geqslant \beta) \leqslant e^{-\left(\frac{\beta^2}{\sigma^2} - \frac{\beta^2}{2\sigma^2}\right)} = e^{-\frac{\beta^2}{2\sigma^2}} \longrightarrow ②$$

☆ **Exc:** Tail bounds for a Gaussian: $Z \sim N(0,1)$, for $t > 0$

$$\left(\frac{1}{t} - \frac{1}{t^3}\right)e^{-t^2/2} \leqslant \mathbb{P}(Z \geqslant t) \leqslant \frac{1}{t}e^{-t^2/2}$$

③ **Constructing sG rv's out of old:**

Claim: Let $X_i \in sG(\sigma_i^2)$ are indep for $i = 1, 2, \ldots, n$. Then $\sum w_i X_i \sim sG$.

Proof:

$$\mathbb{E}e^{s\left(\sum_{i=1}^n w_i X_i\right)} = \prod_{i=1}^n \mathbb{E}e^{s w_i X_i} \leqslant \prod_{i=1}^n e^{s^2 w_i^2 \sigma_i^2/2}$$

$$= \exp\left(\frac{s^2}{2}\sum_{i=1}^n w_i^2\sigma_i^2\right)$$

$$\therefore \sum w_i X_i \sim sG\left(\sum w_i^2 \sigma_i^2\right). \quad \square$$

④ <u>Hoeffding's Inequality:</u> Say $X_i \in sG(\sigma_i^2)$ be indep random variables and thus $\sum X_i \sim sG(\sum \sigma_i^2)$.

$$\therefore \quad \boxed{\mathbb{P}\left(\sum_{i=1}^{n} (X_i - \mathbb{E}(X_i)) \geq t \right) \leq \exp\left(- \frac{t^2}{2 \sum \sigma_i^2} \right)} \quad \square$$

[Proof ③ + ②]

⑤ <u>Hoeffding's Lemma:</u> All bounded rv's are subGaussian.

<u>Q1.</u> Say $\text{Ran}(X) \subseteq [a,b]$. Then $\text{Var}(X) \leq \frac{1}{4}(b-a)^2$.

<u>Proof:</u>

$$\left(X - \frac{a+b}{2} \right)^2 \leq \frac{1}{4}(b-a)^2 \quad \text{wp } 1$$

$$\mathbb{E}\left(X - \frac{a+b}{2} \right)^2 \leq \frac{1}{4}(b-a)^2$$

$$X = \begin{cases} a & \text{wp } \frac{1}{2} \\ b & \text{wp } \frac{1}{2} \end{cases}.$$

$$\text{Var}(X) = \min_c \mathbb{E}(X-c)^2 \leq \frac{1}{4}(b-a)^2. \quad \blacksquare$$

✩ <u>Change of measure:</u>

$$\begin{tikzcd} (\Omega, \mathcal{F}, \mathbb{P}) \arrow[r, "\text{use}"', "f"] \arrow[d, "X"'] & (\Omega, \mathcal{F}, \tilde{\mathbb{P}}) \arrow[dl, "X"] \\ \mathbb{R} & \end{tikzcd}$$

<u>Discrete:</u> X, pmf $(p_i)_{i \in I}$, and let $f \geq 0$, $f : I \to [0,\infty)$ s.t. $f_i = f(i)$

$\sum_{i \in I} f(i)\, p_i$ $\qquad I \subseteq \mathbb{R}$.

Then define $q_i = \dfrac{f(i)\, p_i}{\sum f(i)\, p_i} \rightsquigarrow$ Gives another pmf.

<u>Cont:-</u> X rv, $: \Omega \to \mathbb{R}$, p - pdf, and say $f : \Omega \longrightarrow \mathbb{R}$ s.t. $\mathbb{E}f = \int_{\mathbb{R}} f(x) p(x)\, dx < \infty$, $f \geq 0$

Then, $q(x) = \dfrac{f(x)\, p(x)}{\mathbb{E}f} \longrightarrow$ Another pdf.

Used another random variable f to give a new measure for X.
[Change of Measure / Tilting]

Proof of Hoeffding's Ineq: (WLOG) $X \in [a,b]$ a.s. and $\mathbb{E}X = 0$

- $\psi_X(s) = \log M_X(s)$ (cumulant generating fn, log-mgf)

i) $\psi_X(0) = \log \mathbb{E}\, e^0 = 0$

ii)
$$\psi_X'(s) = \frac{d}{ds} \log \mathbb{E} e^{sX} = \frac{\frac{d}{ds}\mathbb{E} e^{sX}}{\mathbb{E} e^{sX}} = \frac{1}{\mathbb{E} e^{sX}}\, \mathbb{E}\left[\frac{\partial}{\partial s} e^{sX}\right]$$
$$= \frac{\mathbb{E}[X e^{sX}]}{\mathbb{E}[e^{sX}]} = \mathbb{E}\left[X \frac{e^{sX}}{\mathbb{E} e^{sX}}\right]$$

$\psi_X'(0) = 0$

iii)
$$\psi_X''(s) = \frac{1}{[\mathbb{E} e^{sX}]^2}\left(\mathbb{E}[e^{sX}]\, \mathbb{E}[X^2 e^{sX}] - (\mathbb{E}[X e^{sX}])^2\right)$$
$$= \mathbb{E}\left[X^2 \frac{e^{sX}}{\mathbb{E} e^{sX}}\right] - \left(\mathbb{E}\left[X \frac{e^{sX}}{\mathbb{E} e^{sX}}\right]\right)^2 .$$

Exponential Tilting

$$= \mathbb{E}_q(X^2) - (\mathbb{E}_q(X))^2$$, where q is the new pdf of X.

$$\mathbb{E}\left[X^2 \frac{e^{sX}}{\mathbb{E} e^{sX}}\right] = \int_{\mathbb{R}} x^2 \underbrace{\left(\left(\frac{e^{sx}}{\mathbb{E} e^{sX}}\right) \cdot p(x)\right)}_{q(x)} dx = \int_{\mathbb{R}} x^2 \underbrace{q(x)}_{\text{new pdf}}\, dx$$

$X \in [a,b]$

$$\psi_X''(s) = \mathrm{Var}_q(X) \leq \frac{1}{4}(b-a)^2 \longrightarrow (*)$$

∴ Integrating twice,
$$\int_0^s \int_0^{s_1} \psi_X''(s_2)\, ds_2\, ds_1 \leq \frac{1}{4}(b-a)^2 \underbrace{\int_0^s \int_0^{s_1} ds_2\, ds_1}_{=\frac{1}{2}}$$

$$\therefore \quad \psi_X(s) \leq \frac{1}{8}(b-a)^2 s^2 \qquad \square$$

$$\Rightarrow M_X(s) \leq e^{\frac{1}{2} s^2 \left(\frac{1}{4}(b-a)^2\right)}$$

[MDP − Symmetrization]

i.e., $X \in sG\left(\frac{1}{4}(b-a)^2\right)$. $\square$

(Hoeffding's Ineq.)

Thm: Let $X_i \in [a_i, b_i]$ be indep rv's. Then

$$\mathbb{P}\left(\sum (X_i - \mathbb{E}X_i) \geq \beta \right) \leq \exp\left(\frac{-2\beta^2}{\sum (b_i - a_i)^2} \right) \to \circledast$$

$$\mathbb{P}\left(\sum (X_i - \mathbb{E}X_i) \leq -\beta \right) \leq \exp\left(\frac{-2\beta^2}{\sum (b_i - a_i)^2} \right)$$

$$\therefore \quad \mathbb{P}\left(\left| \sum X_i - \mathbb{E}X_i \right| \geq \beta \right) \leq 2 \exp\left(\frac{-2\beta^2}{\sum (b_i - a_i)^2} \right) \quad \boxtimes$$

eg. X_i iid $\mathrm{Ber}_{\pm}\left(\frac{1}{2}\right)$. $S_n = \sum_{i=1}^n X_i$ — Repr a simple random walk in 1D.

$$\mathbb{P}\left(\sum_{i=1}^n X_i \geq c\sqrt{n} \right) \overset{\leq \frac{1}{c^2}}{\leq} \exp\left(- \frac{2 (c\sqrt{n})^2}{4n} \right) = e^{-\frac{c^2}{2}}. \longrightarrow \sum_{i=1}^n X_i = O_p(\sqrt{n})$$

$$\boxed{\sum_{i=1}^n X_i \in [-c\sqrt{n}, c\sqrt{n}] \text{ whp.}}$$

$\mathbb{P}\left(\sum X_i \geq c n^{\frac{1}{2}+\varepsilon} \right) \leq$ ← Much sharper using Hoeffding ineq. $\square$

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3. Subexponential Random Variables:

① **Def$^{\underline{n}}$:** $X \sim s\mathcal{E}(\nu, \alpha)$ if $\mathbb{E} e^{\lambda(X - \mathbb{E}X)} \leq \exp\left(\frac{\lambda^2 \nu}{2} \right) \ \forall \ |\lambda| \leq \frac{1}{\alpha}$.

eg. $X \sim \mathrm{Exp}(1)$. Then,

$$\mathbb{E} e^{sX} = \int_0^\infty e^{sx} \cdot e^{-x} \, dx = \lambda \left(-\frac{e^{-(1-s)}}{1-s} \Big|_0^\infty \right) = \begin{cases} \frac{1}{1-s} & \text{if } s < 1 \\ \infty & s > 1 \end{cases}$$

② Tail Probabilities: $X \sim s\mathcal{E}(\nu,\alpha)$

$$\mathbb{P}(X - \mathbb{E}X \geq \beta) \leq \begin{cases} e^{-\frac{\beta^2}{2\nu}} & \text{if } 0<\beta \leq \frac{\nu}{\alpha} \\ e^{-\frac{\beta}{2\nu}} & \text{if } \beta \geq \frac{\nu}{\alpha}. \end{cases}$$

③ Sums: $X_i \sim s\mathcal{E}(\nu_i,\alpha_i)$, then $\sum w_i X_i \sim s\mathcal{E}\left(\sum w_i^2 \nu_i,\ \max_{i=1}^{n} |w_i|\alpha_i\right)$

④ Bernstein's Inequality: $X_i \sim s\mathcal{G}(\nu_i, \alpha)$

$$\mathbb{P}\left(\left|\sum (X_i - \mathbb{E}X_i)\right| \geq \underline{\underline{\beta}}\right) \leq \begin{cases} 2e^{-\frac{\beta^2}{2\sum w_i^2 \nu_i}} & \text{if } 0<\beta\leq\frac{\nu}{\alpha} \quad \text{(Moderate deviations)} \\ 2e^{-\frac{\beta}{2\alpha}} & \text{if } \beta \geq \frac{\nu}{\alpha}. \quad \text{(Large deviations)} \end{cases}$$

⑤ Result: Let $|X| \leq \underline{\underline{c}}$ a.s. and $\mathbb{E}X = \mu$, $\operatorname{Var} X = \nu$. Then $X \sim s\mathcal{E}(2\nu, \underline{\underline{2c}})$ $\longrightarrow$ [Reading H.W.]

(Bernstein Ineq.)

Thm: If $|X_i| \leq c \ \forall\, i = 1(1)n$ and X_i's are indep, then

$$\mathbb{P}\left(\sum_{i=1}^{n}(X_i - \mathbb{E}X_i) \geq \beta\right) \leq \begin{cases} \exp\left(-\frac{\beta^2}{2\underline{\underline{\sum_{i=1}^{n}\operatorname{Var}(X_i)}}}\right) & \text{if } 0<\beta\leq \frac{\sum \operatorname{Var}(X_i)}{2c} \\ \exp\left(-\frac{\beta}{4c}\right) & \text{if } \beta \geq \frac{\sum \operatorname{Var}(X_i)}{2c} \end{cases}$$

$$\mathbb{P}\left(\sum (X_i - \mathbb{E}X_i) \geq \beta\right) \leq \exp\left(\frac{-\beta^2}{2\left[\frac{1}{4}\sum (b_i - a_i)^2\right]}\right) \longrightarrow ⊛$$

> Upper bd to $\operatorname{var}(X_i)$

1) For large β, Hoeffding is better.

$\underline{\underline{p = o(1)}}$

2) For moderate β: eg. $Y \sim \operatorname{Ber}(p)$. $\operatorname{Var}(Y) = p(1-p) \downarrow$

$(b-a)^2 = 1$

[Reading H.W. – Example 2.4.18 (Max degree of ERRG's) MDP]

Pg – 78.

MDP: Modern Discrete Prob – Sebastian Roch.

$p = \frac{\log n}{n}$

eg. X_i iid $\mathrm{Ber}(p)$, and let β be moderate

Hoeff: $\mathbb{P}\left(\sum (X_i - \mathbb{E} X_i) \geq \beta \right) \leq \exp\left(- \frac{2\beta^2}{n} \right)$

Bern: " $\leq \exp\left(- \frac{\beta^2}{2np(1-p)} \right) = \exp\left(- \frac{\beta^2}{2\log n} \right)$ ✓

Q: X_i iid $N(0,1)$ and $\mathbb{E} \sup_{i=1}^{n} X_i$ - estimate.

- $\mathbb{E} \sup_{i=1}^{n} X_i \leq C \sqrt{\log n}$ ✓ extends to sG rvs

- $\mathbb{E} \sup_{i=1}^{n} X_i = x^* \sqrt{\log n} - \frac{3}{2x^*} \frac{\log \log n}{\sqrt{\log n}} + O\left(\frac{1}{\sqrt{\log n}} \right)$.

Special to $N(0,1)$.