HDP Session - 5   2/06/25

Martingales

Recall: 1. Martingales : $X_0, X_1, X_2, \dots \longrightarrow M_0, M_1, M_2, \dots$ $(M_t)_{t\geq 0}$ – Martingale

$$\mathbb{E}[M_{t+1} \mid X_0, \dots, X_t] = M_t.$$

2. SuperM – $\mathbb{E}[M_{t+1} \mid X_0, \dots, X_t] \leq M_t$ $\longrightarrow$ (Casinos).

SubM – $\mathbb{E}[M_{t+1} \mid X_0, \dots, X_t] \geq M_t$

3. Predictable processes : $H_t \dashv X_0, \dots, X_{t-1}$.

τ stopping time : $\{\tau \leq t\}, \{\tau = t\} \dashv X_0, \dots, X_t$

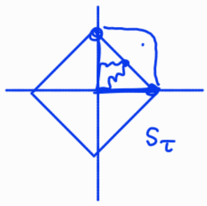
$a \wedge b = \min\{a, b\}$

☆ **Stopped Process:** $(M_n)_n$ is a adapted process, τ is a stopping time, then $(M_n^\tau)_n$ $[M_n^\tau = M_{n\wedge\tau}]$ is called the Stopped process for M_n^τ.

- $(X_0, \dots, X_t, \dots) \longrightarrow M_0, M_1, \dots$, $M_t \dashv X_0, \dots, X_t$

generating process

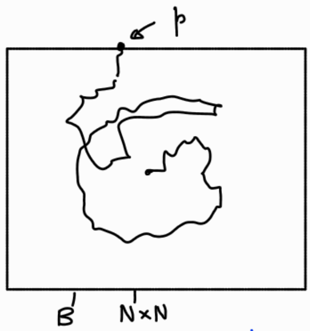
"(M_n) is adapted to the process $(X_n)_n$"



- $S_0, S_1, S_2, \dots$
- Stopped process:

$S_0, S_1, \dots, \underbrace{S_n = p}_{S_\tau}, p, p, p \dots$

- τ = Hitting time for B
- S_n – Random Walk
- S_n^τ = Stopped process



$\rightarrow$ Corresponding to a process S_n and a ST τ, we can define

$$(S_\tau)(\omega) = S_{\tau(\omega)}(\omega) \quad \forall \omega \in \Omega.$$

- $S_\tau(\omega) \in B \quad \forall \omega \in \Omega$.

$\rightarrow$ S_τ can be defined on the set $\{\tau < \infty\}$. In our case $\mathbb{P}\{\tau < \infty\} = 1$

$\therefore$ S_τ can be defined a.s.

Stopped martingales: $\underset{=}{(M_t^\tau)_t} = (M_{\tau\wedge t})_t$

$\exists\, (X_n)_n \xleftarrow{\text{wrt}} \tau$

$(M_n) \xrightarrow{\text{wrt}}$

1. **Thm:** If M is a super M, then $(M_{t\wedge\tau})_t$ is also a super M.

Proof: $\mathbb{E}[M_{t\wedge\tau} \mid \overbrace{X_0, X_1, \dots, X_{t-1}}^{\mathcal{F}_{t-1}}] = M_{(t-1)\wedge\tau}$

$$M_{t\wedge\tau} = M_t \mathbb{1}_{[\tau \geq t]} + \sum_{i=0}^{t-1} M_i \mathbb{1}_{\tau=i}$$

$$\mathbb{E}[M_{t\wedge\tau} \mid \mathcal{F}_{t-1}] = \mathbb{E}[M_t \underbrace{\mathbb{1}_{[\tau\geq t]}}_{} \mid \underline{\underline{\mathcal{F}_{t-1}}}] + \sum_{i=0}^{t-1} \mathbb{E}[\underbrace{M_i \mathbb{1}_{\tau=i}}_{} \mid \mathcal{F}_{t-1}]$$

$$= \mathbb{E}[M_t \mid \mathcal{F}_{t-1}] \mathbb{1}_{\tau\geq t} + \sum_{i=0}^{t-1} M_i \mathbb{1}_{\tau=i}$$

$$\leq M_{t-1} \mathbb{1}_{\tau \geq t} + \sum_{i=0}^{t-1} M_i \underline{\underline{\mathbb{1}}}_{\tau=i}$$

$$= M_{t-1} \mathbb{1}_{\tau \geq t-1} + \sum_{i=0}^{t-2} M_i \mathbb{1}_{\tau=i} = M_{(t-1)\wedge\tau}$$

2. M_t super M, then $\mathbb{E} M_t \leq \mathbb{E} M_t^\tau \leq \mathbb{E} M_0$.

Proof:

$$M_{\tau\wedge t} - M_0 = \sum_{i=1}^{t} (M_i - M_{i-1}) \mathbb{1}_{(\tau \geq i)}$$

$$M_t - M_{\tau\wedge t} = \sum_{i=1}^{t} (M_i - M_{i-1}) \mathbb{1}_{(\tau < i)}$$

Q: What about $\mathbb{E} M_\tau$?

- $t\wedge\tau(\omega) \uparrow \tau(\omega)$ as $t \to \infty$

$$\lim_{t\to\infty} M_{t\wedge\tau}(\omega) = \lim_{t\to\infty} M_{\underset{\to \tau(\omega)}{t\wedge\tau(\omega)}}(\omega)$$

$$= M_{\tau(\omega)}(\omega)$$

- $\lim_{t\to\infty} M_{t\wedge\tau} = M_\tau$ pointwise a.e. $\quad$ ($\tau < \infty$ a.e.)

$\to \quad \mathbb{E} M_{t\wedge\tau} \leq \mathbb{E} M_0 \quad \Rightarrow \quad \lim_{t\to\infty} \mathbb{E}[M_{t\wedge\tau}] \leq \mathbb{E} M_0$

$\overset{?}{\Rightarrow} \quad \underline{\mathbb{E} M_\tau \leq \mathbb{E} M_0}$

?

(Optional Stopping Theorem)

Three Theorems:

1. Monotone Convergence Thm: (MCT) If $X_n \uparrow X$ (i.e., $\forall \omega$ (a.e. almost), $X_n(\omega) \uparrow X(\omega)$)

then $\lim\limits_{n\to\infty} \mathbb{E} X_n = \mathbb{E}\left[\lim\limits_{n\to\infty} X_n\right] = \mathbb{E}X$

$(\mathbb{E}X_1 \underline{\underline{>}} -\infty)$

$\lim_{n\to\infty} \mathbb{E}X_n$: limit of $\mathbb{R}$-no.s

$\lim_{n\to\infty} X_n$: pointwise (a.e.) limit of RVs.

If $\mathbb{E}X = +\infty$, then $\lim\limits_{n\to\infty} \mathbb{E}X_n = +\infty$.

Almost everywhere: $(\Omega, \mathbb{P}) \xrightarrow{X} \mathbb{R}$, $X(\omega)$ is defined $\forall \omega$

$X(\omega)$ is defined $\forall\ \omega \in \Omega' \subseteq \Omega$

$\mathbb{P}(\Omega \setminus \Omega') = 0.$

2. Dominated Conv. Thm (DCT): Say $X_n \longrightarrow X$ ptwise a.e., $\exists\ Y \geq 0$ s.t. $|X_n| \leq Y \ \forall n \in \mathbb{N}$ and $\mathbb{E}\,\underline{\underline{Y}} < \infty$. Then

$$\lim_{n\to\infty} \mathbb{E}X_n = \mathbb{E}\left[\lim_{n\to\infty} X_n\right] = \mathbb{E}X$$

3. Fatou's Lemma: Say $(X_n)_n$ is a sequence of +ve rvs.

$$\mathbb{E}\left[\liminf_{n\to\infty} X_n\right] \leq \liminf_{n\to\infty} \mathbb{E}[X_n]$$

$$\begin{tikzcd} \boxed{FL} \arrow[r, bend left] & \boxed{MCT} \arrow[l, bend left] \arrow[r] & \boxed{DCT} \end{tikzcd}$$

(☆) Thm (Optional Stopping Theorem): (M_n) is a superM. $\boxed{\mathbb{E}M_\tau \leq \mathbb{E}M_0}$ if one of the the following holds:

✓ i) $\tau \leq T$ a.s. for some $T \in \mathbb{N}_0$.

→ ii) $\mathbb{E}\tau < \infty$ and $|M_n| \leq B \ \forall n \in \mathbb{N}_0$.

iii) $|M_n - M_{n-1}| \leq c$ and $\tau < \infty$ a.s.

If M_n is a Martingale, then $\Rightarrow \mathbb{E}M_\tau = \mathbb{E}M_0$.

☆ iv) $M_n \geq 0 \ \forall n \in \mathbb{N}$ (Prove directly using Fatou's Lemma.)

Proof of (ii): $M_{\tau \wedge t} = \sum_{i=0}^{t} M_i \mathbb{1}_{\tau = i} \xrightarrow[t \to \infty]{} M_\tau$ (p-a.e.)

DCT: $|M_{t \wedge \tau}| \leq \sum_{i=0}^{t} |M_i| \mathbb{1}_{\tau=i} \leq B \sum_{i=0}^{t} \mathbb{1}_{\tau=i} = B[(\tau \wedge t)+1]$

$(\tau \wedge t) \leq \tau \Rightarrow \mathbb{E}\left(B((\tau \wedge t)+1)\right) \leq B(1+\mathbb{E}\tau) < \infty$ as $\mathbb{E}\tau < \infty$.

$\therefore$ Justified ✓

(iii): $M_{t \wedge \tau} = M_0 + \sum_{i=0}^{t} (M_i - M_{i-1}) \mathbb{1}_{\tau \geq i}$ ... ✓

$\sum (a_n - a_{n-1}) b_n$
$\wr$
$\sum a_m (b_{m+1} - b_m)$

(☆) Applications : Betting Strategies:

- A stock has price M_n at time n. $(X_0, X_1, \dots)$
- We buy H_n quantities at time n. ← Predictable process.
- W_n – winning on n^{th} day.

$$\boxed{W_n = W_0 + \sum H_n (M_n - M_{n-1})}.$$

Thm: If M_n is a Martingale. Then W_n is also a M.

↪ • $\mathbb{E} W_n = \mathbb{E} W_0$

In a Casino, M_n is a super M and $H_n \geq 0 \ \forall n$.

$\longrightarrow$ $\mathbb{E} W_n \leq \mathbb{E} W_0$

$\underline{\mathbb{E} W_\tau \leq \mathbb{E} W_0}$ $(\tau \leq T)$

- Say (M_n) is a Martingale and (H_n) be any pred. process. Then

$(H \cdot M)_n := \sum_{i=0}^{n} H_i (M_i - M_{i-1}) \longrightarrow$ Discrete Integral wrt the martingale $(M_n)_n$.

$\int H(t)\, dM_t \longrightarrow$ Stochastic Integrals

☆ **Exercise**: Let $X_1, X_2, \ldots$ iid X, $\mathbb{E}|X| < \infty$, $\mathbb{E}X = \mu$. Let $S_n = \sum_1^n X_i$.

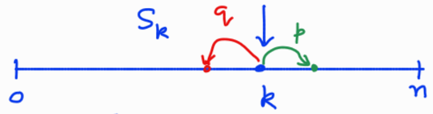
i) $\mathbb{E}(S_\tau) = \mu \, \mathbb{E}\tau$ for τ s.t. $\mathbb{E}\tau < \infty$

ii) $\mathbb{E}(S_\tau^2) = \sigma^2 \, \mathbb{E}\tau$ if $\mathbb{E}X^2 = \sigma^2$ and $\mathbb{E}X = 0$.

(Wald's Identities)

Exc: For Gambler's Ruin etc.

H.W.



$\mathbb{P}(\text{Ruin}) = \mathbb{P}(S_\tau = 0)$ $\qquad$ $\mathbb{E}[\tau_{end}] = \tau_{\{0,n\}}$

☆ **Thm (Convergence)**: M_t is a super M and $M_t \geq 0$ $\forall t$. Then $\exists M_\infty$ s.t. $\lim M_t = M_\infty$ a.s. and

$$\mathbb{E}M_\infty \leq \lim_{t \to \infty} \mathbb{E}M_t \leq \mathbb{E}M_0$$

eg: If M_n is subM and ϕ is a convex $f^{\underline{n}}$, then $\phi(M_n)$ is a subM.

⇝ We can get a good handle on M_τ.

☆ **Doob Decomposition**:

Thm: Let $(Y_n)_n$ be an adapted stochastic process. Then $Y_n = M_n + A_n$, where M_n is a Martingale and A_n is predictable.

Proof:

$$Y_n = Y_0 + \underbrace{\sum_{t=1}^n \left(Y_t - \mathbb{E}(Y_t \mid \mathcal{F}_{t-1})\right)}_{\text{Martingale}} + \underbrace{\sum_{t=1}^n \left(\mathbb{E}(Y_t \mid \mathcal{F}_{t-1}) - Y_{t-1}\right)}_{\text{Predictable.}}$$ ▨

If Y_n is a subM — ↑
super M — ↓

• $Y_n = W_n + A_n$

h

$z \longmapsto \mathbb{E}(M_\tau^z)$

Harmonic $f^{\underline{n}}$s

eg. M_n a martingale, $Y_n = M_n^2$

$$\underset{\text{predictable}}{A_n} = \sum_{t=1}^{n}\left(\mathbb{E}[M_t^2 \mid \mathcal{F}_{t-1}] - M_{t-1}^2\right) = \sum_{t=1}^{n}\left(\mathbb{E}[M_t^2 \mid \mathcal{F}_t] - 2M_{t-1}\mathbb{E}[M_t \mid \mathcal{F}_t] + M_{t-1}^2\right)$$

$$= \sum_{t=1}^{n} \mathbb{E}\left[(M_t - M_{t-1})^2 \mid \mathcal{F}_{t-1}\right] \longrightarrow \text{Square Variation of the martingale } (M_t)$$

In general, $f \in C^2(\mathbb{R})$ and M_t is a martingale

$$\underset{\text{(Ito's formula)}}{f(M_t)} = \underbrace{f(M_0) + \int_0^t f'(M_s)\, dM_s}_{\text{Martingale}} + \underbrace{\frac{1}{2}\int_0^t f''(M_s)\, d\langle M\rangle_s}_{\text{Predictable}}$$

§. Back to Conc. Inequalities:

1. Doob's submartingale Ineq: $(M_n)_{n\in\mathbb{N}_0}^{\geq 0}$ is a subM. Then for $a>0$

$$\mathbb{P}\Big(\overbrace{\sup_{1\leq s\leq t} M_s \geq a}^{M_\tau \geq a}\Big) \leq \frac{\mathbb{E}M_t}{a}$$

Comment: Using Markov's Ineq. $\sup_{0\leq s\leq t} \mathbb{P}(M_s \geq a) \leq \frac{\mathbb{E}M_t}{a}$

Proof: Define $\tau = \inf\{s \geq 1 \mid M_s \geq a\} \wedge t$

$$\mathbb{P}\Big(\sup_{1\leq s\leq t} M_s \geq a\Big) = \mathbb{P}(M_\tau \geq a) \leq \frac{\mathbb{E}M_\tau}{a} \leq \frac{\mathbb{E}M_t}{a} \quad \blacksquare$$

⊘ $S_n = \sum X_i$, X_i iid $\text{Ber}_\pm\left(\frac{1}{2}\right)$

$$\mathbb{P}(|S_n| > c\sqrt{n}) \lesssim 2e^{-\frac{c^2}{2}} \quad ; \quad \mathbb{P}(|S_n| > c\sqrt{n}) \leq \frac{2}{c}$$

$$\mathbb{P}\Big(\max_{0\leq t\leq n} |S_t| > c\sqrt{n}\Big) \leq \frac{2}{c} \quad (*).$$

2. Kolmogorov's Max$^{\underline{l}}$ Ineq: M_n – subM, $\mathbb{E}M_0 = 0$

$$\mathbb{P}\left(\max_{1 \le s \le t} |M_s| > a \right) \leqslant \frac{Var(M_t)}{a^2}.$$

<u>Proof</u>: (Use Doob's Inequality).

Hoeffding $\rightsquigarrow$ Azuma-Hoeffding.

X_n iid X

$\underline{\underline{\max |S_n|}}$ <u>not</u> $\max |X_n|$ → Very diff answer. $\square$

- 9/06 – { 45-50 mins → ...
  Rest → Present$^{\underline{n}}$ by Saptashwa

- 12/06 – → (*)

– 2 weeks of Holidays –