HDP Session 7

09/06/2025

Supremum: $\circ \ \sup_{t\in T} X_t$ → T is finite; → T is infinite

§. Finite Supremum $\quad (\mathbb{E}X_t = 0)$

$$\sup_{t\in T} X_t \leq \sum_{t\in T} |X_t|$$

$$\therefore \quad \mathbb{E}\left(\sup_{t\in T} X_t\right) \leq |T| \sup_{t\in T} \underline{\mathbb{E}|X_t|}$$

'Entropy factor' to bring the sup out of the expectation.

Better: $(p \geq 1)$ ↓ Assuming p^{th} moment exist.

$$\left(\mathbb{E} \sup_{t\in T} X_t\right) \leq \left(\mathbb{E} \sup_{t\in T} |X_t|^p\right)^{1/p} \quad \left[\text{Jensen: } z \longmapsto z^p \text{ is a convex } f^n\right]$$

$$\leq \left(\mathbb{E}\left(\sum_{t\in T} |X_t|^p\right)\right)^{1/p}$$

$$\leq \underline{|T|^{1/p}} \sup_{t\in T} \underbrace{\left(\mathbb{E}|X_t|^p\right)^{1/p}}_{\|X_t\|_p} \longrightarrow (1)$$

What about $X_t \in sG(\sigma^2)$?

For $sG(\sigma^2)$, we know $\left(\mathbb{E}|X|^p\right)^{1/p} \leq c\|X\|_{\psi_2}\sqrt{p}$

Using this, we have

$$\mathbb{E}\left(\sup X_t\right) \leq \left(c \sup_{t\in T} \|X_t\|_{\psi_2}\right) \underline{|T|^{1/p}\sqrt{p}} \qquad \forall\, p \geq 1$$

Optimize: $f(p) = p^k n^{1/p}$ for $p \geq 1$, $n \geq 1$

$$\frac{k}{p} - \frac{\log n}{p^2} = 0 \quad \Rightarrow \quad p = \frac{\log n}{k}$$

$$\therefore \quad \min_p f(p) = \left(\frac{\log n}{k}\right)^k \cdot n^{\frac{k}{\log n}} = \underline{\frac{1}{k^k}} e^k (\log n)^k \quad \blacksquare$$

$\therefore$ For sG rvs: $\quad \mathbb{E}\left[\sup X_t\right] \leq \left(c \sup_{t\in T} \|X_t\|_{\psi_2}\right) \underline{\sqrt{\log|T|}}$

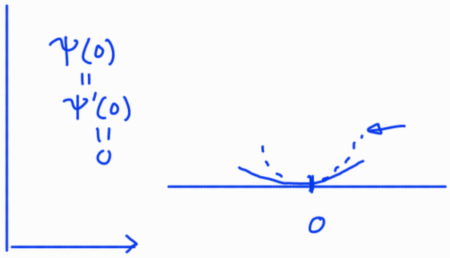
basically optimal entropic factor.

Exc: $X \sim s\mathcal{G} \iff X^2 \sim s\mathcal{E}$.

See Vershyin

For $s\mathcal{E}$ rvs: $\left(\mathbb{E}[|X|^p]\right)^{1/p} = \|X\|_p \leq \left(c \|X\|_{\psi_1}\right) p$

$$\therefore \quad \mathbb{E} \sup X_t \leq \left(c \sup_{t \in T} \|X_t\|_{\psi_1}\right) \log|T|. \longrightarrow$$



Mgf and cgf: $\psi_X(\lambda) = \log \mathbb{E} e^{\lambda X}$

$$\psi_X^*(z) = \sup_{\lambda \in \mathbb{R}} \left(\lambda z - \psi(\lambda)\right)$$

If we take $\mathbb{E} X = 0$ (i.e., $\psi_X'(0) = 0$), then how does $\psi_X^*(z)$ look like?

$$\psi_X'(\lambda) = \frac{\mathbb{E}[X e^{\lambda X}]}{\mathbb{E}[e^{\lambda X}]}$$

$$\psi_X^*(z) = \lambda^* z - \psi(\lambda^*), \quad \text{s.t.} \quad \psi'(\lambda^*) = z$$

$$\Rightarrow \lambda^* = (\psi')^{-1}(z) > 0 \quad \text{as } z > 0$$

For $z > 0$, $\psi_X^*(z) = \sup_{\lambda > 0} \left(\lambda z - \psi_X(\lambda)\right)$.

$$\Rightarrow \boxed{\frac{\psi_X^*(z) + \psi_X(\lambda)}{\lambda} \geq z \quad \forall\ z > 0,\ \lambda > 0 \quad \text{with equality at } \lambda^* = (\psi')^{-1}(z)}$$

① **Lemma:** ($\mathbb{E}$-max$^{\underline{l}}$ inequality) Let $(X_t)_{t \in T}$ be a collection of rvs s.t. $\log \mathbb{E}(e^{\lambda X_t}) \leq \psi(\lambda)$ $\forall \lambda > 0$ s.t. $\psi(0) = 0$, $\psi'(0) = 0$.

Then,
$$\mathbb{E}\left[\sup (X_t)\right] \leq (\psi^*)^{-1}(\log|T|).$$

Proof: $\mathbb{E}\left[\sup_t (X_t)\right] \leq \frac{1}{\lambda} \log \mathbb{E}\left(\sup_t e^{\lambda X_t}\right)$ [Jensen]

$$\leq \frac{1}{\lambda} \log \left(\sum_{t \in T} \mathbb{E}\, e^{\lambda X_t}\right)$$

$$\leq \frac{1}{\lambda} \log\left(|T|\, \psi(\lambda)\right) = \frac{\log|T| + \psi(\lambda)}{\lambda}$$

Take z s.t. $\psi^*(z) = \log|T|$ and $\lambda = \lambda^* = (\psi^*)^{-1}(z)$
$= \log|T|$.

Then we have

Tight of indep rvs.

$$\boxed{\mathbb{E}\left(\sup X_t\right) \leqslant (\psi_X^*)^{-1}\left(\log|T|\right)}$$

$\square$

eg. For $X_t \in sG(\sigma^2)$, we have $\psi_X(\lambda) = \frac{\lambda^2\sigma^2}{2}$ works. Thus,

$$\mathbb{E}\left(\sup X_t\right) \leqslant \sqrt{2\sigma^2 \log|T|}.$$

(2) Lemma (Max$^{\underline{l}}$ tail inequality) [Same hyp as in Lemma-(1)]

$$\mathbb{P}\left[\sup X_t \geqslant (\psi^*)^{-1}\left(\log|T| + u\right)\right] \leqslant e^{-u}$$

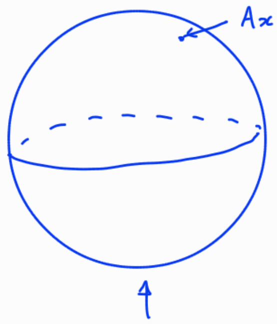
Proof: Complete this. $\square$

Example: A – $n\times n$ random matrix, $A = (a_{ij})_{n\times n}$, $a_{ij} \overset{\text{iid}}{\sim} \text{Ber}_{\pm}\left(\frac{1}{2}\right)$.
Then $\mathbb{E}\,\|A\|_{op} = \; ?$,

$$\|A\|_{op} = \sup_{\|x\|=1} \|Ax\| = \sup_{x\in S^{n-1}} \|Ax\|$$

- $\|Ax - Ay\| \leqslant \|A\|_{op}\, \|x-y\|$

$\uparrow$ modulus of cont.
is a random variable.

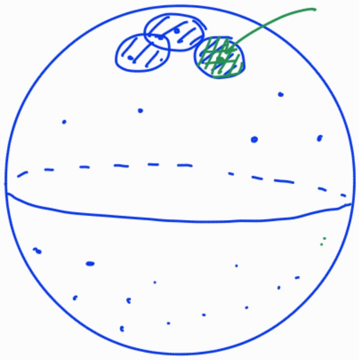


☆ Def$^{\underline{n}}$: A SP (X_t) is called Lipschitz with M.O.C. C if

$$|X_t - X_s| \leqslant C\, d(s,t)$$

[T is a metric space]

<u>Covering Method:</u>



T

$X_x = \|Ax\|$

$X_y = \|Ay\|$

1. To find an ε-net $\mathcal{N}_\varepsilon$ for all $\varepsilon > 0$, i.e., a <u>finite</u> set $\mathcal{N}_\varepsilon \subseteq T$ s.t. $\forall\, t \in T$, $\exists\, s \in \mathcal{N}_\varepsilon$ s.t. $d(s,t) < \varepsilon$.

   In other words, $T \subseteq \bigcup_{s \in \mathcal{N}_\varepsilon} B_\varepsilon(s)$.

(1) can be carried out iff T is compact.

2. $\sup_{s \in \mathcal{N}_\varepsilon} X_s \longrightarrow$ Can be tackled as before.

3. <u>Lemma:</u>

$$\mathbb{E}\left[\sup_{t \in T} X_t\right] \leq \varepsilon\, \mathbb{E}[C] + \mathbb{E} \sup_{s \in \mathcal{N}_\varepsilon} X_s$$

$$\leq \varepsilon\, \mathbb{E}[C] + (\psi_X^*)^{-1}\left(\log |\mathcal{N}_\varepsilon|\right)$$

In particular, for the $sG(\sigma^2)$ case, we have

$$\boxed{\mathbb{E}\left[\sup_{t \in T} X_t\right] \leq \varepsilon\, \mathbb{E}[C] + \sqrt{2\sigma^2 \log |\mathcal{N}_\varepsilon|}}$$

<u>Proof:</u> Consider $\mathcal{N}_\varepsilon$ an ε-net of T. Then $\forall\, t \in T$, $\exists\, s = s(t) \in \mathcal{N}_\varepsilon$ s.t. $d(t,s) \leq \varepsilon$. Then,

$$X_t \leq X_s + |X_t - X_s| \leq C\varepsilon + \sup_{s \in \mathcal{N}_\varepsilon} X_s$$

Taking $\sup_t$, we are done. $\square$