Session 9

Finite Maxima:

1. $X_t \sim sG(\sigma^2) \quad \forall\ t \in T$, then $\boxed{\mathbb{E} X_t = 0}$

$\longrightarrow \mathbb{E}\left[\sup_{t\in T} X_t\right] \leq \sqrt{2\sigma^2 \log |T|}$ $\qquad X_t = \mu + Z_t$

$\sqrt{\log|T|} \longrightarrow$ Entropic factor.

$\longrightarrow \mathbb{P}\left[\sup_{t\in T} X_t > \sqrt{2\sigma^2 \log|T|} + u\right] \leq e^{-\frac{u^2}{2\sigma^2}}$

2. (T,d) a compact metric space. If $(X_t)_{t\in T}$ is a Lipschitz process

$|X_s - X_t| \leq \underset{\sim}{C}\, d(s,t)$ a.s. $\forall s,t \in T$ $\longleftarrow$ $\boxed{X_t \sim sG(\sigma^2)}$

then (Covering method)

$$\mathbb{E}\left[\sup_{t\in T} X_t\right] \leq \mathbb{E}\left[\sup_{t\in T} X_{\pi(t)} + \sup_{t\in T}\left(X_t - X_{\pi(t)}\right)\right]$$

$$= \mathbb{E}\left[\sup_{t\in T} X_{\pi(t)}\right] + \mathbb{E}\left[\sup_{t\in T}\left(X_t - X_{\pi(t)}\right)\right]$$

$|t - \pi(t)| < \varepsilon$

finite supremum $\leq \sqrt{2\sigma^2 \log N(T,d,\varepsilon)}$

infinite supremum $\leq \varepsilon\, \mathbb{E}[C]$.

3. What if we do not have Lipschitz bounds? — Chaining!

Say $\varepsilon_1, \varepsilon_2, \dots \downarrow 0$, say $\mathcal{N}_k$ — ε_k net (with mapping f$^{\underline{n}}$ π_k).

Say $|\mathcal{N}_k| = N_k = N(T, d, \varepsilon_k)$.

$$\mathbb{E}\left[\sup_{t\in T} X_t\right] \leq \mathbb{E}\left[\sup_{t\in T} X_{\pi_1(t)}\right] + \mathbb{E}\left[\sup_{t\in T}(X_t - X_{\pi_1(t)})\right]$$

$$\leq \mathbb{E}\left[\,\cdot\,\right] + \mathbb{E}\left[\sup_{t\in T}\left(X_{\pi_2(t)} - X_{\pi_1(t)}\right)\right] + \mathbb{E}\left[\sup\left(X_t - X_{\pi_2(t)}\right)\right]$$

Can be bdd w/out LC.

finite supremum.

$$\leq \mathbb{E}\left[\sup_{t\in T} X_{\pi_0(t)}\right] + \sum_{i=1}^{n} \mathbb{E}\left[\sup_{t\in T}\left(X_{\pi_i(t)} - X_{\pi_{i-1}(t)}\right)\right]$$

$$+ \mathbb{E}\left[\sup_{t\in T}\left(X_t - X_{\pi_n(t)}\right)\right]$$

(☆) $\downarrow 0$ as $n\to\infty$

If (☆) is true, then

$$\mathbb{E}\left[\sup_{t\in T} X_t\right] \leq \mathbb{E}\left[\sup_{t\in T} X_{\underbrace{\pi_0(t)}}\right] + \sum_{i=1}^{\infty} \mathbb{E}\left[\sup_{t\in T}\left(\underbrace{X_{\pi_i(t)} - X_{\pi_{i-1}(t)}}\right)\right]$$

$\varepsilon_0 > diam(T)$

<u>Reductions</u>: Let $\varepsilon_k = 2^{-k}$ ... decreasing seq. of ε-Net sizes.

$$\boxed{\mathbb{E}\left[\sup_{t\in T} X_t\right] \leq \sum_{i\in\mathbb{Z}} \mathbb{E}\left[\sup_{t\in T}\left(\underbrace{X_{\pi_i(t)} - X_{\pi_{i-1}(t)}}_{sG.}\right)\right] + \mathbb{E}\left[X_{t_0}\right]}$$

$\boxed{\text{Chaining}}$

(☆) Sub Gaussian Process: A process (X_t) on (T,d) is called a SGP if

$$① — \mathbb{E}\left[e^{\lambda(X_s - X_t)}\right] \leq e^{\lambda^2 d(s,t)^2/2} \quad \forall\, s,t\in T \text{ and } \lambda\in\mathbb{R}.$$

$\longrightarrow$ Automatically: $\mathbb{E}X_s = \mathbb{E}X_t \;\forall\, s,t$.

WLOG, $\mathbb{E}X_t = 0 \;\forall\, t\in T$.

In prob. version of L.C.

Weakening of LC: $|X_s - X_t| \leq C_{s,t}\, d(s,t)$

where $C_{s,t}$ is a subgaussian (1) r.v.

$$\iff \mathbb{P}\left(|X_s - X_t| > x\, d(s,t)\right) \leq 2e^{-x^2/2} \iff ① \text{ holds.}$$

☆ Separable Process: (T,d) - m.s. and (X_t) is a process on it. It is called separable if $\exists$ a countable set $T_0 \subseteq T$ s.t

$$\mathbb{P}\left\{\omega : X_t(\omega) \in \lim_{\substack{s \to t \\ \text{along } T_0}} X_s(\omega)\right\} = 1$$

$$\Leftrightarrow \quad X_t \in \lim_{\substack{s \to t \\ \text{along } T_0}} X_s$$

Sort of a continuity statement

Thm: (Dudley's Ineq) For a SSGP, (★) holds.

$$\mathbb{E}\left[\sup_{t \in T} X_t\right] \leq \sum_{i \in \mathbb{Z}} \mathbb{E}\left[\sup_{t \in T}\left(X_{\pi_i(t)} - X_{\pi_{i-1}(t)}\right)\right]$$

Now, Card. of i^{th} term $\leq N_i \cdot N_{i-1} \leq N(T, d, 2^{-i})$

Subgaussian moment of $X_{\pi_i(t)} - X_{\pi_{i-1}(t)}$ is $d(\pi_i(t), \pi_{i-1}(t))$

$$d(\pi_i(t), \pi_{i-1}(t)) \leq d(t, \pi_i(t)) + d(t, \pi_{i-1}(t))$$
$$\leq 2^{-i} + 2^{-i+1} \leq 3 \times 2^{-i}$$

$$\therefore \quad \mathbb{E}\left[\sup_{t \in T}\left(X_{\pi_i(t)} - X_{\pi_{i-1}(t)}\right)\right] \leq 3 \times 2^{-i} \times \sqrt{2 \log\left(N(T,d,2^{-i})^2\right)}$$
$$= 6 \times 2^{-i} \sqrt{\log N(T,d,2^{-i})}$$

(I)

$$\therefore \quad \boxed{\mathbb{E}\left[\sup_{t \in T} X_i\right] \leq 6 \times \underbrace{\sum_{i \in \mathbb{Z}} 2^{-i} \sqrt{\log N(T,d,2^{-i})}}_{\text{depends only on } (T,d)}}$$

Dudley's Inequality.

$$\int \sqrt{\log \ldots} \leq \sum 2^{-i} \sqrt{\log |\ldots} \leq 2 \int \sqrt{\log \ldots}$$

Nothing

$$\sum_{i\in\mathbb{Z}} 2^{-i}\sqrt{\log N(T,d,2^{-i})} \leq 2\sum_{i\in\mathbb{Z}}\int_{2^{-i-1}}^{2^{-i}} \sqrt{\log N(T,d,2^{-i})}\,d\varepsilon$$

$$\leq 2\sum_{i\in\mathbb{Z}}\int_{2^{-i-1}}^{2^{-i}} \sqrt{\log N(T,d,\varepsilon)}\,d\varepsilon$$

$$= 2\int_0^\infty \sqrt{\log N(T,d,\varepsilon)}\,d\varepsilon$$

$$\boxed{\mathbb{E}\Big[\sup_{t\in T} X_t\Big] \leq 12\underbrace{\int_0^{\cancel{\infty}\ \mathrm{diam}(T)} \sqrt{\log N(T,d,\varepsilon)}\,d\varepsilon}_{\text{Entropic factor}}} \quad (\mathrm{I})$$

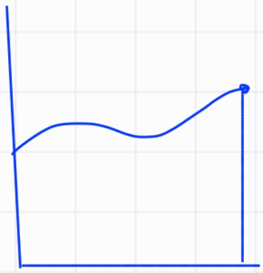
---

3. Monte Carlo Integration:

- $X \sim \mathrm{Unif}[0,1]$, $X_1, X_2, X_3, \ldots$ iid X.
- f is a cont. bdd f^{n} on $[0,1]$, $f(X_i)$ iid $f(X)$

$$\frac{f(X_1)+\cdots+f(X_n)}{n} \xrightarrow{p} \int_0^1 f(x)\,dx$$

(MCI)



- X_i iid μ,

$$\frac{1}{n}\sum_{i=1}^n f(X_i) \xrightarrow{p} \int f\,d\mu$$

$$X\sim\mu \equiv \mathbb{P}(X\in A) =: \mu(A)$$

$$\mathbb{E}\left|\frac{1}{n}\sum_{i=1}^n f(X_i) - \mathbb{E}f(X)\right| \leq \mathbb{E}\left|\frac{1}{n}\sum_{i=1}^n \big(f(X_i)-\mathbb{E}f(X_i)\big)\right|$$

$$\leq \left[\mathbb{E}\left(\frac{1}{n^2}\left(\sum_{i=1}^n \big(f(X_i)-\mathbb{E}f(X_i)\big)\right)^2\right)\right]^{1/2}$$

$$\leq \left[\frac{1}{n^{\cancel{2}}}\,\cancel{n}\cdot \mathrm{Var}\big(f(X)\big)\right]^{1/2} \leq \frac{\|f\|_\infty}{\sqrt{n}}$$

Convergence has rate $\frac{1}{\sqrt{n}}$.

Uniform bound: $\mathbb{E} \sup_{f\in\mathcal{F}} \underbrace{\left(\frac{1}{n}\sum f(X_i) - \mathbb{E} f(X) \right)}_{X_f}$

i) Consider $|f(x) - f(y)| \leq |x-y|$.

ii) $f \longmapsto f + c$, $X_f = X_{f+c}$. So, $0 \leq f \leq 1$.

$$\mathcal{F} = \left\{ f \in \mathrm{Lip}([0,1]) \;\middle|\; 0 \leq f \leq 1,\ \mathrm{Lip}(f) \leq 1 \right\}.$$

1. Covering method:

$\mathbb{E}\left(\sup_f X_f \right)$ → Lipschitz bound, → Covering no.

$$|X_f - X_g| = \left| \frac{1}{n}\sum(f(X_i) - \mu_f) - \frac{1}{n}\sum g(X_i) - \mu_g \right| \leq \underbrace{2}\, \|f-g\|_\infty$$

$$e^{c'/\varepsilon} \lesssim \underset{=}{N(\mathcal{F}, \|\cdot\|_\infty, \varepsilon)} \leq \underset{=}{e^{c/\varepsilon}}$$

sG moment $\|X_f\|_{\psi_2}^2 \leq \frac{1}{n}$.

$$X_f = \sum_{i=1}^n \underbrace{\frac{(f(X_i) - \mathbb{E} f(X_i))}{n}}_{\uparrow\ [-\frac{1}{n}, \frac{1}{n}]} \rightsquigarrow \|X_f\|_{\psi_2}^2 \leq \kappa \cdot \frac{1}{n^2}$$

$\neq$

$a \leq Y \leq b$

$\|Y\|_{\psi_2} \leq \left(\frac{b-a}{2}\right)^2$

[Hoeffding's Lemma]

$$\therefore \quad \mathbb{E}\left[\sup_{f\in\mathcal{F}} X_f \right] \leq 2\varepsilon + \sqrt{\frac{2}{n} \log N(\mathcal{F}, \|\cdot\|, \varepsilon)}$$

$$= 2\varepsilon + \sqrt{\frac{2c}{\varepsilon n}} \quad \longleftarrow \text{optimum at } \varepsilon \sim n^{-1/3}$$

$$\boxed{\begin{aligned} &\mathbb{E}\left[\sup_{f\in\mathcal{F}} X_f \right] \lesssim n^{-1/3} \\ &\mathbb{E}\, |X_f| \lesssim \underset{=}{n^{-1/2}} \end{aligned}} \longrightarrow \text{Not optimum}$$

Heuristically,

L.C.: $\mathbb{E}\, |X_f - X_g| \leq \left(\mathbb{E}\, |X_f - X_g|^2 \right)^{1/2}$

$$= \left[\mathbb{E} \left(\frac{1}{n}\sum_{i=1}^n \left((f(X_i) - \mu_f - \frac{1}{n} (g(X_i) - \mu_g \right)\right)^2 \right]^{1/2}$$

$$\lesssim \frac{1}{\sqrt{n}} \|f-g\|_\infty \qquad 2\|f-g\|_\infty$$

$$\mathbb{E}\left[\sup_{t\in T} X_t\right] \leq \frac{\varepsilon}{\sqrt{n}} + \sqrt{\frac{2c}{\varepsilon n}} \lesssim \frac{1}{\sqrt{n}}$$

This points to the fact that our bound is indeed suboptimum.

<u>Chaining Method :</u>

$$\| X_f - X_g \|_{\psi_2} \leq \frac{1}{\sqrt{n}} \| f - g \|_\infty$$

$$\mathbb{E}\left[e^{\lambda(X_f - X_g)}\right] \leq e^{\lambda^2 \frac{\|f-g\|_\infty^2}{n} / 2}$$

$\therefore$ (X_f) is a SSGP on $\left(\mathcal{F}, \frac{\|\cdot\|_\infty}{\sqrt{n}}\right)$.

Dudley's ineq :

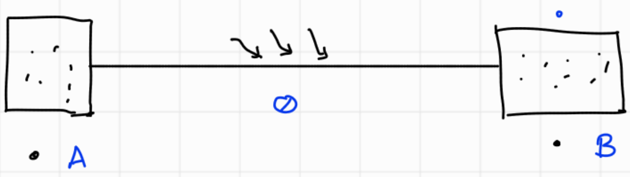
$$\mathbb{E}\left[\sup_t X_t\right] \leq 12 \int_0^\infty \sqrt{\log N\left(\mathcal{F}, \frac{\|\cdot\|_\infty}{\sqrt{n}}, \varepsilon\right)}\, d\varepsilon$$

$$= 12 \int_0^\infty \sqrt{\log N(\mathcal{F}, \|\cdot\|_\infty, \varepsilon\sqrt{n})}\, d\varepsilon$$

$$= \frac{12}{\sqrt{n}} \int_0^1 \sqrt{\log N(\mathcal{F}, \|\cdot\|_\infty, \varepsilon')}\, d\varepsilon'$$

$$= \frac{12}{\sqrt{n}} \int_0^1 \sqrt{\frac{2c}{\varepsilon}}\, d\varepsilon \leq \frac{c}{\sqrt{n}}. \qquad \blacksquare$$

Error Correcting Codes (ECCs)



abc $\rightsquigarrow$ $k^{(=3)}$

$\underbrace{abc \;\; abc \;\; abc \; \ldots \; abc}_{(2r+1)}$

- Hypothesis – Environment can corrupt upto r bits

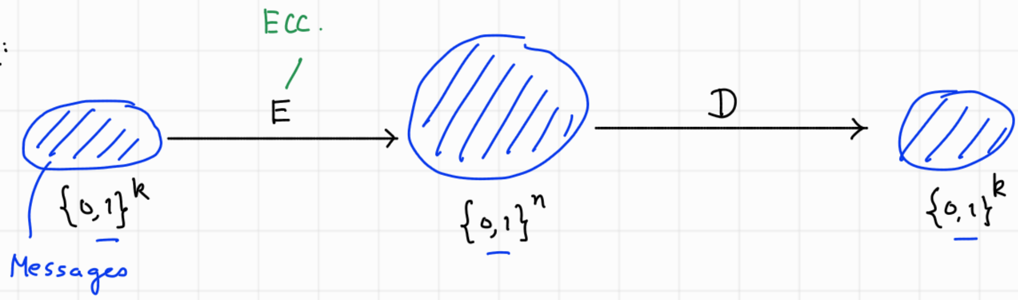
$$n = (2r+1)k$$

$\{E(x) : x \in \{0,1\}^k\}$ – codebook

$\downarrow$

codewords.

★ **Setup:**



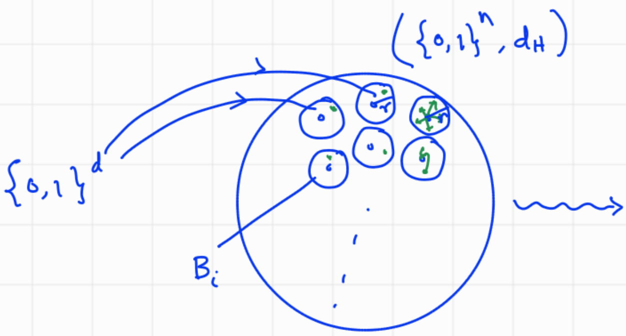
$$D(E(x)) = x$$

Also, $\forall y \in \{0,1\}^n$ s.t. $d_H(y, E(x)) \leq r$, then
$D(y) = x$.

D, E: r-ECC.

- $d_H((1,0,0,1,1), (0,1,0,1,1)) = 2 = \ell^1(\cdot,\cdot)$

One way:



Error correcting Code:

1. Choose n large enough s.t. we can pack $\{0,1\}^d$ balls of radius r inside $\{0,1\}^n$.
2. If $E(x) \in B_i$, then $\mathcal{N}(E(x)) \in B_i$ for that unique i.

   $D \longrightarrow E(x)$

Question: How big is n, for given k, r? (Hamming)

Required condition: $\boxed{P(\{0,1\}^n, d_H, 2r) \geqslant 2^k}$

Thm:

$$\frac{2^n}{\sum_{k=0}^{m}\binom{n}{k}} \leqslant N(\{0,1\}^n, d_H, m) \leqslant P(\{0,1\}^n, d_H, m) \leqslant \frac{2^n}{\sum_{k=0}^{\lfloor m/2\rfloor}\binom{n}{k}}$$

Proof: $(\{0,1\}^n, d_H)$: $(\cdot, \cdot, \ldots, \cdot)$

$$|B_r(\underline{x})| = \sum_{k=0}^{r}\binom{n}{k}$$

The rest is obvious. ▨

Exc:

$$\left(\frac{n}{m}\right)^m \leqslant \binom{n}{m} \leqslant \sum_{k=0}^{m}\binom{n}{k} \leqslant \left(\frac{en}{m}\right)^m.$$

Suff cond$^{\underline{n}}$ for ECC: $\Leftarrow \quad \dfrac{2^n}{\sum_{k=0}^{2r}\binom{n}{k}} \geqslant 2^k$

$$\Leftarrow \quad n-k \geqslant \log_2\left(\sum_{k=0}^{2r}\binom{n}{k}\right)$$

$$\Leftarrow \quad n-k \geqslant \log_2\left(\left(\frac{en}{2r}\right)^{2r}\right) = 2r\log_2\left(\frac{en}{2r}\right).$$

$\therefore$ Suff cond$^{\underline{n}}$ $\boxed{n-k \sim r\log\left(\frac{n}{r}\right)}$ — Optimum

$n = (2k+1)r$ — Horrible.