Worksheet - 1

(Moment Methods, SubGaussian and SubExponential RVs)

Q. Exercises:

1. (Paley-Zygmund Inequality)

   a) Let $X \geq 0$ a.s. be a random variable s.t. $\mathbb{E}X^2 < \infty$. Then show that
   $$\mathbb{P}(X \geq \lambda \mathbb{E}X) \geq (1-\lambda^2)\frac{(\mathbb{E}X)^2}{\mathbb{E}(X^2)} \quad \forall\ \lambda \in [0,1]$$

   b) This is an analogue of Paley-Zygmund Inequality for higher moments. Let $p \in (1,\infty)$ and $1/p + 1/q = 1$, and $X \geq 0$ a.s. be a random variable s.t. $\mathbb{E}X^{2p} < \infty$. Then
   $$\mathbb{P}(X \geq \lambda) \geq \frac{(\mathbb{E}X^2 - \lambda^2)^q}{(\mathbb{E}(X^{2p}))^{q/p}} \quad \forall\ 0 \leq \lambda \leq \sqrt{\mathbb{E}(X^2)}$$

2. For any $\varepsilon > 0$, find a random variable X s.t. for X_n iid X, $\mathbb{E}X < \infty$ and constants $c, C > 0$ s.t.
   $$\frac{c}{n^\varepsilon} \leq \mathbb{P}\left(\sum_{i=1}^{n}(X_i - \mathbb{E}X_i) \geq 1\right) \leq \frac{C}{n^\varepsilon}$$

So random variables may concentrate only very slowly about the mean.

Problems from Vershynin's book (2^{nd} edition):

- 2.2, 2.3, 2.4, 2.6 (on tails of Normal Random Variables)
- 2.23, 2.26, 2.27, 2.28 (Characterization of sG random variables)

There are many more exercises on Subgaussian and Subexponential properties, precise descriptions of subgaussian norms, and Orlicz norms. Feel free to try those out as well.

These topics marked in green, although instructive, are optional.

8. Problems:

Problems from Vershynin:

- 2.38, 2.39 (maxima of subgaussian random variables)
- Prop$^{\underline{n}}$ - 2.5.1, 2.18, 2.19 (on degrees of random graphs)

Problems from MDP - Sebastian Roch:

- 2.18, 2.19 (Poisson Convergence & application)
- 2.5, 2.6 (Pairwise independence $\not\Rightarrow$ Exponential Concentration)

Reading Assignments (from MDP)

Section - 2.3.2

Section - 2.3.3 - The latter parts on computing Branching no.s may be skipped or skimmed through

Section - 2.2.4 - The proof of contour lemma may be skipped
Also try out the corresponding exercises 2.3 & 2.4 at the end of the chapter.