Worksheet - 2

∅ Topics: Everything from Moment Methods to Martingales.

Exercises:

MDP : 3.4, 3.5, 3.6 (Routine calculations)
3.7 (An alternate proof of Azuma-Hoeffding Inequality)

1) (Bennett's Inequality) Let $X_1, \ldots, X_N$ be independent random variables and $|X_i - \mathbb{E}X_i| \leq K$ $\forall i$. Then show that

$$\mathbb{P}\left(\sum_{i=1}^{N} (X_i - \mathbb{E}X_i) \geq \beta \right) \leq \exp\left(-\frac{\sigma^2}{K^2} h\left(\frac{K\beta}{\sigma^2} \right) \right) \quad \forall \beta > 0,$$

where $\sigma^2 = \sum_{i=1}^{n} \mathrm{Var}(X_i)$ and $h(u) = (1+u)\log(1+u) - u$.

Note that in the small deviation regime $u \ll 1$, $h(u) \approx u^2$ and thus it gives us a sG decay. Again in the large deviation regime say $u \geq 2$, $h(u) \approx \frac{1}{2} u \log u$, which gives a sE bound. In that sense, it is analogous to Bernstein's Inequality.

2. (Khintchine's Inequality) Let $X_1, \ldots, X_N$ be independent sG random variables with mean zero and variance one.

Let $\underset{\sim}{a} = (a_1, a_2, \ldots, a_N)^t \in \mathbb{R}^N$. Then prove that for every $p \in [2, \infty)$, we have

$$\|\underset{\sim}{a}\|_2 \leq \mathbb{E}\left[\langle \underset{\sim}{a}, \underset{\sim}{X} \rangle^p \right]^{1/p} \leq C K \sqrt{p}\, \|\underset{\sim}{a}\|_2$$

where $C > 0$ is an absolute constant, and $K = \max_i \|X\|_{\psi_2}$.

This also has versions for $p = 1$ and $p \in (1,2)$. For more details, see Vershynin's book.

Problems:

van Handel's HDP: Try out problem 3.7, very instructive. It demonstrates that variance proxy could be vastly incorrect. We'll later see ways to sharpen this using the Entropy Method.

Also try problems - 3.8, 3.9. A Hilbert Space is an inner product space where the induced metric is complete. You may replace 'Hilbert Space' by $\mathbb{R}^N$ for arbitrary $N \in \mathbb{N}$ for this problem.

1. (Percolation) i) Let $G = (V, E)$ be a locally-finite (i.e., each degree is finite) infinite graph on a countable vertex set. For $v \in V$, let $N(v,n)$ be the number of self-avoiding paths starting from v. We define the connective constant of G as

$$\kappa(G) = \sup_{v \in V} \left(\limsup_{n \to \infty} N(v,n)^{1/n} \right)$$

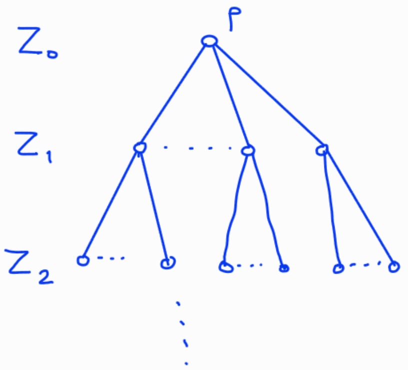
Show that $d \leq \kappa(\mathbb{Z}^d) \leq 2d-1$. Again, if G has maximum degree Δ, show that $\kappa(G) \leq \Delta - 1$. Can the equality be achieved?

ii) Consider G as above with $\kappa(G) < \infty$. Let $G(p)$ be the random graph obtained by deleting edges in G independently with probability $(1-p)$. Let C_v denote the connected component of $v \in V$. Then show that for $p < \kappa^{-1}$,

$$\mathbb{P}(|C_v| = \infty) = 0$$

Hint: Note that $\mathbb{P}(|C_v| = \infty) = \lim_{n \to \infty} \mathbb{P}\left(\begin{array}{c}\exists \text{ a self-avoiding path of length } n \\ \text{starting at } v\end{array}\right)$ and use (i).

2. (Galton Watson tree)



Let ξ be a $\mathbb{N}_0$-valued random variable with a pmf $(p_k)_{k\in\mathbb{N}_0}$.
Let $\mu = \mathbb{E}\xi$ and $\sigma^2 = \mathrm{Var}(\xi)$.
We generate a random tree as follows:
Let $X_{i,t} \overset{iid}{\sim} \xi$ for $i,t \geq 0$. We start from the root ρ, which has $X_{0,0}$ children. In this first generation, the i^{th} child proceeds to have $X_{i,1}$ children. This process continues to give us the random tree.

In particular, if $(Z_t)_t$ denotes the number of children in the t^{th} generation, then $Z_0 := 1$ and $Z_t = \sum_{i=1}^{Z_{t-1}} X_{i,t}$. We are interested in observing whether the process survives forever or dies out at a finite time.

i) Using first and second moment methods, show that

$$\mathbb{P}(\text{Extinction}) = \lim_{n\to\infty} \mathbb{P}(Z_n = 0) = \begin{cases} 1 & \text{if } \mu < 1 \\ c_\mu > 0 & \text{if } \mu > 1 \end{cases}$$

ii) We can perform a more intricate analysis using the generating function method. To avoid trivialities, let $p_0 > 0$, $p_0 + p_1 < 1$.

a) Consider the probability generating function of ξ : $f(s) = \mathbb{E}[s^\xi] = \sum_{k=0}^{\infty} s^k p_k$.
Show that pgf of Z_n is $f\circ f\circ\cdots\circ f$ (n times), where '$\circ$' indicates composition. [Hint: Condition appropriately]

b) Noting that f is strictly convex, show that

$$\mathbb{P}(\text{Extinction}) = \lim_{n\to\infty} \underbrace{(f\circ f\circ\cdots\circ f)}_{n \text{ times}}(0) = \begin{cases} 1 & \text{if } \mu \leq 1 \\ c_\mu & \text{if } \mu > 1 \end{cases},$$

where $0 < c_\mu < 1$ is the smallest root of the equation $f(s) = s$.

iii) Prove that $M_n := \dfrac{Z_n}{\mu^n}$ is a martingale, and in particular $M_n \to M_\infty$ a.s.
Show that $\mathbb{P}(M_\infty = 0) = 1$ or c_μ.

3. Let $G(n,p)$ be the Erdös Renyi graph with $p = \frac{\lambda}{n}$, $\lambda \in (0,\infty)$ and n large. Set $I(n,p)$ = No. of isolated vertices in $G(n,p)$. Show that

$$\frac{I(n,p)}{n} \xrightarrow{a.s.} e^{-\lambda}$$

Note that this is a SLLN type result — the empirical average no. of isolated vertices converges to the probability that a vertex is isolated. The events however are not independent.

(Show convergence in probability and then use B.C. Lemma to extend it is convergence a.s..)

4. (Random Geometric Graphs) Let $\mathcal{X} = [0,1]^2$ and define the torroidal metric on $\mathcal{X}$ by

$$d(x,y) := \min\{|x-y+z| : z \in \mathbb{Z}^2\}$$

Let $X_1, \dots, X_n$ be chosen u.a.r. from $\mathcal{X}$. Construct a graph G with $V(G) = \{1, 2, \dots, n\}$ and

$$E(G) = \{\{i,j\} \mid |X_i - X_j| \leq r_n\},$$

where the critical radius $r_n \in (0,\infty)$ is chosen so that

$$(n-1)\pi r_n^2 = \lambda \in (0,\infty) \text{ fixed.}$$

Such graphs, constructed out of the underlying metric structure of a given space, are called Random Geometric Graphs (RGGs).

a) Show that $\deg(i) \xrightarrow{d} \mathrm{Poi}(\lambda)$ as $n \to \infty$.

b) If I_n be the no. of isolated vertices, then $n^{-1} I_n \xrightarrow{a.s.} e^{-\lambda}$.

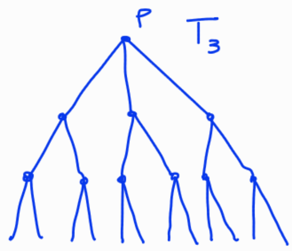
5. (Polya's Urn Model) One green and one red ball are placed in an urn at $t=0$. At all subsequent times, a ball is drawn from the urn, put back, and another ball of the same colour is added to the urn. This process in continued forever.

i) Let G_t be the no. of green balls at time t. Show that

$$\mathbb{P}(G_t = m+1) = \frac{1}{t+1}$$

ii) Show that $M_t = \frac{G_t}{t+2}$ is a martingale. Thus, $M_t \xrightarrow{a.s.} M_\infty$. Find the limiting distribution M_∞.

6. (Percolation on trees) Consider a d-regular tree T_d for $d \geq 3$. We designate each edge of the tree T_d to be open wp p and closed wp $(1-p)$ for some $p \in [0,1]$.



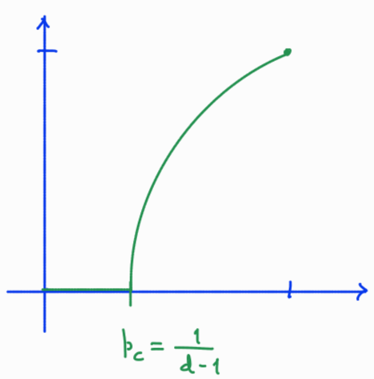
This gives us a random tree $\mathbb{T}_d$ with edges coloured 'open' or 'closed'. Let ρ denote the root of $\mathbb{T}_d$ and C_ρ the connected component of ρ formed only by open edges (called the open cluster of ρ).

a) Derive a polynomial equation for $\theta(p) = \mathbb{P}_p(|C_\rho| = +\infty)$ using the symmetry of the tree $\mathbb{T}_d$.

b) Show that
$$\theta(p) = \begin{cases} = 0 & \text{if } p \leq \frac{1}{d-1} \\ > 0 & \text{if } p > \frac{1}{d-1} \end{cases}$$
Give an explicit characterisation of $\theta(p)$ for $p \in [0,1]$. Use Q2 for this. For $d = 3$, compute $\theta(p)$ explicitly.

c) Show that $\theta(p) = 0$ for $p < \frac{1}{d-1}$ follows from Q1 as well.

d) (Extra) Show that $\theta(p)$ is monotonic for $p \in [0,1]$.



The threshold $p_c = \frac{1}{d-1}$ the critical probability for percolation.

e) Show that $\mathbb{P}_p(\mathbb{T}_d \text{ has an infinite open cluster}) = \begin{cases} 0 & \text{if } p \leq p_c \\ 1 & \text{if } p > p_c \end{cases}$.