Worksheet - 3

∅ Martingales

1. (SLLN for Martingale Differences) We argued that the martingale difference sequence $(M_n - M_{n-1})_{n \in \mathbb{N}}$ behaved somewhat like an iid sequence — on the grounds that they are jointly uncorrelated and have similar concentration properties. In this problem, we obtain a SLLN result for $(M_n - M_{n-1})_{n \in \mathbb{N}}$, where M_n is a martingale. Let $N_t := M_t - M_{t-1}$ for brevity.

   i) We assume that $(M_n)_n$ satisfies the condition that $\sum_{t \geq 1} \frac{\mathbb{E}(N_t^2)}{t^2} < \infty$. In particular, $H_n := \sum_{t=1}^{n} \frac{N_t}{t}$ is an L^2 martingale.

   Use Kolmogorov's maximal inequality for submartingales to show that $\forall \, \varepsilon > 0$,
   $$\mathbb{P}\left(\inf_{n \in \mathbb{N}} \sup_{m \geq n} |H_m - H_n| \geqslant \varepsilon \right) = 0$$
   In other words $(H_n)_n$ is a Cauchy sequence a.s., and thus converges a.s..

   ii) (Krönecker's Lemma) Consider $(x_n) \subseteq \mathbb{R}$ s.t. $\sum_{n=1}^{\infty} x_n = s$, and a sequence $(b_n) \subseteq [0, \infty)$ s.t. $b_n \uparrow \infty$. Then show that
   $$\frac{1}{b_n} \sum_{t=1}^{n} b_t x_t \longrightarrow 0 \quad \text{as} \quad n \to \infty$$

   iii) Using (i) and (ii), show that for martingales satisfying the assumption in (i), we have
   $$\frac{1}{n} \sum_{t=1}^{n} N_t \xrightarrow{a.s.} 0$$

   iv) Taking X_i iid X, $\mathbb{E}X = \mu$, $\mathbb{E}X^2 < \infty$, we have a proof of SLLN for iid sequences. This was the first proof of SLLN, due to A.N. Kolmogorov.

v) Show that the condition in (i) is satisfied if $\mathbb{E}(M_t^2) = O(t^{2-\varepsilon})$ for any $\varepsilon > 0$.

vi) The argument in (i) really shows something stronger. If W_n is an L^2 martingale, then W_n converges a.s. It also converges in L^2, i.e., $\exists\, W_\infty$ s.t. $\mathbb{E} W_\infty^2 < \infty$ and

$$\mathbb{E}\left[(W_\infty - W_n)^2\right] \longrightarrow 0 \quad \text{as} \quad n \longrightarrow \infty$$

Moreover $\lim_{n \to \infty} \mathbb{E} W_n^2 = \mathbb{E} W_\infty^2$.

2. (Petersburg Game) We agreed in class that betting strategies are futile, they can never beat a fair game. Consider the following situation:

Let X_i iid $\mathrm{Ber}_{\pm}(\frac{1}{2})$ be the generating process, and $M_n = \sum_{i=1}^{n} X_i$ be the martingale. Recall that we require a predictable process $(H_t)_t$ and then our winnings are

$$W_n = \sum_{t=1}^{n} H_t (M_t - M_{t-1}) = \sum_{t=1}^{n} H_t X_t$$

We set $H_0 = $ Rs 1. If we win $(X_0 = 1)$, then we stop the game ($H_t = 0$ henceforth). Else, we continue betting Rs. $2, 4, 8, \ldots, 2^t$ at times $2, 3, \ldots, t$ till we get $X_t = 1$. Then we stop the game. In all the above cases, we win Rs 1. Does this beat the the system?

i) As $W_n \leq 1$ always, $W_n \longrightarrow W_\infty$ a.s. What is the distribution of W_∞? Note that $\lim_{n \to \infty} \mathbb{E} W_n < \mathbb{E} W_\infty$. This is an example where of random variable which converge a.s. but not in L^1.

ii) Let $\tau = \inf\{t \geq 0 \mid X_t = 1\}$. Then $\mathbb{E} W_0 < \mathbb{E} W_\tau$. Check that conditions (i), (ii), (iii) of Doob's optional stopping theorem fail, while (iv) holds for $(-W_n)_n$.

iii) Let I_n be the amount of money invested at time n. Then $\mathbb{E} I_n = \infty$. This is the problem with this betting strategy — we need an infinite cash reserve, and an infinite time, to win.

Check that $H_n = \min\{ 2^t \mathbb{1}_{[t \le \tau]}, B \}$ gives us $\mathbb{E} W_\tau = 0$ for any cap B.

3. (Random Walk on $\mathbb{Z}$) Let X_i iid $\mathrm{Ber}_\pm(\frac{1}{2})$ and $S_n = \sum_{i=1}^{n} X_i + S_0$. We have seen two martingales corresponding to this — namely the linear and quadratic martingales $L_n = S_n$ and $Q_n = S_n^2 - n$.

i) Show that $W_n := \dfrac{e^{\theta S_n}}{\phi(\theta)^n}$ is a martingale, where $\phi(\theta) = \mathbb{E} e^{\theta X}$.

[Exponential Martingale]

ii) Let $a < x < b$ and $\tau = \inf\{ t \geq 0 \mid S_t \notin (a,b) \}$. Assume that $S_0 := x$. Then, using L_n and Doob's OST, show that

$$\mathbb{P}_x(S_\tau = a) = \frac{b-x}{b-a}, \quad \mathbb{P}_x(S_\tau = b) = \frac{x-a}{b-a}$$

[Here, $\mathbb{P}_x(\cdot) = \mathbb{P}(\cdot \mid S_0 = x)$]

Using Wald's first identity, prove that $\mathbb{E}_x \tau = (b-x)(x-a)$

iii) Fix $S_0 = 0$ and let $\tau_1 = \inf\{ t \geq 0 \mid S_t = 1 \}$

Show that

$$\mathbb{E} s^{\tau_1} = \frac{1 - \sqrt{1-s^2}}{s}$$

Inverting this generating function, show that

$$\mathbb{P}(\tau_0^{(1)} = 2n-1) = \frac{1}{2n-1} \binom{2n}{n} 2^{-2n}$$

iv) Choose $\theta_0 > 0$ s.t. $\phi(\theta_0) = 1$, $S_0 = 0$, $a < 0 < b$ and

$$\tau = \inf\{ t \geq 0 \mid S_t \notin (a,b) \}$$

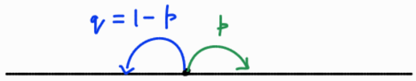
Show that $\mathbb{P}(\tau < \infty) = \exp(-\theta_0 a)$

v) Let $\tau = \inf\{t \geq 0 \mid S_t \notin (-a,a)\}$. Find constants $b, c \in \mathbb{R}$ s.t.

$$Y_n = S_n^4 - 6n S_n^2 + bn^2 + cn$$

is a martingale. Use this to compute $\mathbb{E}\tau^2$.

4. Consider a SRW with $\mathbb{P}(\xi = 1) = p$, $\mathbb{P}(\xi = -1) = q = 1-p$ and ξ_i iid ξ, $S_n = \sum_{i=1}^{n} \xi_i$.



i) Show that for $\varphi(y) := \left(\frac{q}{p}\right)^y$ $\forall y \in \mathbb{Z}$, $\varphi(S_n)$ is a martingale

ii) If $\tau_z = \inf\{t \mid S_t = z\}$ (hitting time of z), then for $a < x < b$, show that

$$\mathbb{P}_x(\tau_a < \tau_b) = \frac{\varphi(b) - \varphi(x)}{\varphi(b) - \varphi(a)}$$

[This gives the ruin probability for the gambler's ruin problem]

Assume $p \in \left(\frac{1}{2}, 1\right)$ for parts (iii) and (iv).

iii) If $a < 0$, then $\mathbb{P}\left(\inf_{t=1}^{\infty} S_t \leq a\right) = \mathbb{P}(\tau_a < \infty) = \left(\frac{q}{p}\right)^a$

iv) If $b > 0$, then $\mathbb{P}(\tau_b < \infty) = 1$ and $\mathbb{E}\tau_b = \frac{b}{2p - 1}$ [This helps us calculate $\mathbb{E}_x \tau_{ruin}$]